Independence of CM points in Elliptic curves

Jacob Tsimerman

University of Toronto

October 8, 2020

Joint work with Jonathan Pila

- $S = X_1(N)$ - Usual modular curve of level N
- $\mathrm{CM} \subset S$ - CM points in S.
- E - elliptic curve over $\overline{\mathbb{Q}}$.
- $\Gamma < E(\overline{\mathbb{Q}})$ - finite-rank if $\Gamma \otimes \mathbb{Q}$ is finite dimensional

- $S = X_1(N)$ - Usual modular curve of level N
- $\mathrm{CM} \subset S$ - CM points in S.
- E - elliptic curve over $\overline{\mathbb{Q}}$.
- $\Gamma < E(\overline{\mathbb{Q}})$ - finite-rank if $\Gamma \otimes \mathbb{Q}$ is finite dimensional

Theorem (Buium-Poonen,2007)

Let S be a modular curve, and $\Phi : S \to E$ a morphism. Let $\Gamma < E(\overline{\mathbb{Q}})$ be finite rank. Then $\phi(CM) \cap \Gamma$ is finite.

- $S = X_1(N)$ - Usual modular curve of level N
- $\mathrm{CM} \subset S$ - CM points in S.
- E - elliptic curve over $\overline{\mathbb{Q}}$.
- $\Gamma < E(\overline{\mathbb{Q}})$ - finite-rank if $\Gamma \otimes \mathbb{Q}$ is finite dimensional

Theorem (Buium-Poonen,2007)

Let S be a modular curve, and $\Phi : S \to E$ a morphism. Let $\Gamma < E(\overline{\mathbb{Q}})$ be finite rank. Then $\phi(CM) \cap \Gamma$ is finite.

If Γ is finitely generated, this follows simply from degree bounds.

2 Generalizations

Theorem (Buium-Poonen,2007)

Let S be a modular curve, , and $V \subset S \times E$ be a correspondence.. Let $\Gamma < E(\overline{\mathbb{Q}})$ be finite rank. Then $\pi_2(\pi_1^{-1}(CM)) \cap \Gamma$ is finite.

2 Generalizations

Theorem (Buium-Poonen,2007)

Let S be a modular curve, , and $V \subset S \times E$ be a correspondence.. Let $\Gamma < E(\overline{\mathbb{Q}})$ be finite rank. Then $\pi_2(\pi_1^{-1}(CM)) \cap \Gamma$ is finite.

- $D/\mathbb{Q}$ -indefinite quaternion algebra
- $X^D(U)$ - Shimura curve parametrizing abelian surfaces with an $\mathcal{O}_D$ action and a U-level structure.

Theorem (Buium-Poonen,2007)

Let $S = X^D(U)$ be a Shimura curve, and $\Phi : S \to E$ a morphism. Let $\Gamma < E(\overline{\mathbb{Q}})$ be finite rank. Then $\phi(CM) \cap \Gamma$ is finite.

2 Generalizations

Theorem (Buium-Poonen,2007)

Let S be a modular curve, , and $V \subset S \times E$ be a correspondence.. Let $\Gamma < E(\overline{\mathbb{Q}})$ be finite rank. Then $\pi_2(\pi_1^{-1}(CM)) \cap \Gamma$ is finite.

- $D/\mathbb{Q}$ -indefinite quaternion algebra
- $X^D(U)$ - Shimura curve parametrizing abelian surfaces with an $\mathcal{O}_D$ action and a U-level structure.

Theorem (Buium-Poonen,2007)

Let $S = X^D(U)$ be a Shimura curve, and $\Phi : S \to E$ a morphism. Let $\Gamma < E(\overline{\mathbb{Q}})$ be finite rank. Then $\phi(CM) \cap \Gamma$ is finite.

Key Ingredient: Galois Equidistribution.

Independence of CM points

Definition

Let $V \subset S \times E$ be a correspondence. Any point in $\pi_2(\pi_1^{-1}(x))$ is called a V-image of x.

Independence of CM points

Definition

Let $V \subset S \times E$ be a correspondence. Any point in $\pi_2(\pi_1^{-1}(x))$ is called a V-image of x.

Definition

Let S be a Shimura curve We say $x_1, x_2 \in CM$ are D-independent if

- *The discriminants $\Delta(x_i)$ are at least D, and*
- *any Shimura subvariety $C \subset S \times S$ containing (x_1, x_2) has degree at least D.*

Independence of CM points

Definition

Let $V \subset S \times E$ be a correspondence. Any point in $\pi_2(\pi_1^{-1}(x))$ is called a V-image of x.

Definition

Let S be a Shimura curve We say $x_1, x_2 \in CM$ are D-independent if

- *The discriminants $\Delta(x_i)$ are at least D, and*
- *any Shimura subvariety $C \subset S \times S$ containing (x_1, x_2) has degree at least D.*

Theorem (Main I, Pila-T, 2019)

Let $V \subset S \times E$ be a correspondence. For any $N > 0$ there exists $D > 0$ such that if $x_1, x_2, \ldots, x_N$ are pairwise D-independent, then any V-images $s_1, \ldots, s_n$ are linearly independent in E.

remarks

- This theorem implies the results of Buium-Poonen, by taking $N = \mathrm{rank}(\Gamma) + 1$.

remarks

- This theorem implies the results of Buium-Poonen, by taking $N = \mathrm{rank}(\Gamma) + 1$.
- The phrasing is a bit careful, since V-images of the same point could be generically dependent.

remarks

- This theorem implies the results of Buium-Poonen, by taking $N = \mathrm{rank}(\Gamma) + 1$.
- The phrasing is a bit careful, since V-images of the same point could be generically dependent.
- The proof relies on large Galois orbits, but not on equidistribution. So S can be any shimura curve:

remarks

- This theorem implies the results of Buium-Poonen, by taking $N = \mathrm{rank}(\Gamma) + 1$.
- The phrasing is a bit careful, since V-images of the same point could be generically dependent.
- The proof relies on large Galois orbits, but not on equidistribution. So S can be any shimura curve:
  - F-totally real
  - B/F - quaternion algebra, split over a single real place $\sigma : F \hookrightarrow \mathbb{R}$.
  - $S = \sigma B(\mathcal{O}_F)\backslash\mathbb{H}$.

remarks

- This theorem implies the results of Buium-Poonen, by taking $N = \mathrm{rank}(\Gamma) + 1$.
- The phrasing is a bit careful, since V-images of the same point could be generically dependent.
- The proof relies on large Galois orbits, but not on equidistribution. So S can be any shimura curve:
  - F-totally real
  - B/F - quaternion algebra, split over a single real place $\sigma : F \hookrightarrow \mathbb{R}$.
  - $S = \sigma B(\mathcal{O}_F)\backslash\mathbb{H}$.
- Instead of looking at CM points, one can instead fix an isogeny class (or hecke orbit) and obtain comparable results.

remarks

- This theorem implies the results of Buium-Poonen, by taking $N = \mathrm{rank}(\Gamma) + 1$.
- The phrasing is a bit careful, since V-images of the same point could be generically dependent.
- The proof relies on large Galois orbits, but not on equidistribution. So S can be any shimura curve:
  - F-totally real
  - B/F - quaternion algebra, split over a single real place $\sigma : F \hookrightarrow \mathbb{R}$.
  - $S = \sigma B(\mathcal{O}_F)\backslash\mathbb{H}$.
- Instead of looking at CM points, one can instead fix an isogeny class (or hecke orbit) and obtain comparable results.
- Tempting but hard to use this to prove large rank of elliptic curves over hilbert class fields, due to the lack of uniformity in $\mathrm{rank}(\Gamma)$.

BP Proof: Galois equidistribution

- K-number field
- For $X/K, x \in X(\overline{\mathbb{Q}})$, $\mu_x := \int_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \delta_{\sigma x}$.

BP Proof: Galois equidistribution

- K-number field
- For $X/K, x \in X(\overline{\mathbb{Q}})$, $\mu_x := \int_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \delta_{\sigma x}$.

Theorem

Equidistribution Results

- *(Duke-Iwaniec) S-quaternionic Shimura curve, $x \in CM$ then $\mu_x \to \mu_{hyp}$.*

BP Proof: Galois equidistribution

- K-number field
- For $X/K, x \in X(\overline{\mathbb{Q}})$, $\mu_x := \int_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \delta_{\sigma x}$.

Theorem

Equidistribution Results

- *(Duke-Iwaniec) S-quaternionic Shimura curve, $x \in CM$ then $\mu_x \to \mu_{hyp}$.*
- *(Zhang) E-elliptic curve, $\Gamma \subset E(\overline{\mathbb{Q}})$ finite rank, $x_i \in \Gamma$. If $\deg(x_i) \to \infty$ then $\mu_{x_i} \to \mu_{haar}$.*

BP Proof: Galois equidistribution

Case I: Shimura Curve

- $\Phi : S \to E, \Phi(s_i) = a_i$

BP Proof: Galois equidistribution

Case I: Shimura Curve

- $\Phi : S \to E, \Phi(s_i) = a_i$
- $\Phi_* \mu_{hyp} = \lim \Phi_* \mu_{s_i} = \lim \mu_{a_i} = \mu_{haar}.$

BP Proof: Galois equidistribution

Case I: Shimura Curve

- $\Phi : S \to E, \Phi(s_i) = a_i$
- $\Phi_* \mu_{hyp} = \lim \Phi_* \mu_{s_i} = \lim \mu_{a_i} = \mu_{haar}$.
- Φ ramified at $s \in S$. Then

$$\epsilon^2 \sim \mu_{haar}(B_\epsilon(\Phi(s))) \gg \mu_{hyp}(B_{\sqrt{\epsilon}}(s)) \gg \epsilon.$$

BP Proof: Galois equidistribution

Case I: Shimura Curve

- $\Phi : S \to E, \Phi(s_i) = a_i$
- $\Phi_* \mu_{hyp} = \lim \Phi_* \mu_{s_i} = \lim \mu_{a_i} = \mu_{haar}$.
- Φ ramified at $s \in S$. Then

$$\epsilon^2 \sim \mu_{haar}(B_\epsilon(\Phi(s))) \gg \mu_{hyp}(B_{\sqrt{\epsilon}}(s)) \gg \epsilon.$$

BP Proof: Galois equidistribution

Case II: Modular Curve

- Cusps have large hyperbolic volume :

$$\mu_{hyp}(q < \epsilon) = \int_{\substack{x \in \mathbb{R}/\mathbb{Z} \\ y > -2\pi \ln \epsilon}} \frac{dxdy}{y^2} = \frac{1}{-2\pi \ln \epsilon}.$$

BP Proof: Galois equidistribution

Case II: Modular Curve

- Cusps have large hyperbolic volume :

$$\mu_{hyp}(q<\epsilon)=\int_{\substack{x\in\mathbb{R}/\mathbb{Z}\\ y>-2\pi\ln\epsilon}}\frac{dxdy}{y^2}=\frac{1}{-2\pi\ln\epsilon}.$$

- $V\subset\ S\times E, x_i\in V, \pi_1(x_i)\in \mathrm{CM}, \pi_2(x_i)\in\Gamma.$

BP Proof: Galois equidistribution

Case II: Modular Curve

- Cusps have large hyperbolic volume :

$$\mu_{hyp}(q < \epsilon) = \int_{\substack{x \in \mathbb{R}/\mathbb{Z} \\ y > -2\pi \ln \epsilon}} \frac{dxdy}{y^2} = \frac{1}{-2\pi \ln \epsilon}.$$

- $V \subset \ S \times E, x_i \in V, \pi_1(x_i) \in \mathrm{CM}, \pi_2(x_i) \in \Gamma.$
- $\mu =$ weak-* limit of μ_{x_i}. Then $\pi_{1*}\mu = \mu_{hyp}, \pi_{2*}\mu = \mu_{haar}$.

BP Proof: Galois equidistribution

Case II: Modular Curve

- Cusps have large hyperbolic volume :

$$\mu_{hyp}(q < \epsilon) = \int_{\substack{x \in \mathbb{R}/\mathbb{Z} \\ y > -2\pi \ln \epsilon}} \frac{dxdy}{y^2} = \frac{1}{-2\pi \ln \epsilon}.$$

- $V \subset \; S \times E, x_i \in V, \pi_1(x_i) \in \mathrm{CM}, \pi_2(x_i) \in \Gamma.$
- $\mu =$ weak-* limit of μ_{x_i}. Then $\pi_{1*}\mu = \mu_{hyp}, \pi_{2*}\mu = \mu_{haar}$.
-

$$\begin{aligned}
(-\ln \epsilon)^{-1} &= \mu_{hyp}(B_\infty(\epsilon)) \\
&\ll \sum_{t \in \pi_1^{-1}(\infty)} \mu(B_t(\sqrt[n]{\epsilon})) \\
&\ll \sum_{t \in \pi_2(\pi_1^{-1}(\infty))} \mu(B_t(\sqrt[n]{\epsilon})) \\
&\ll \sqrt[n]{\epsilon}.
\end{aligned}$$

Unlikely Intersections: Basic Setup

Specials in E^n

- $W \subset E^n$ is *special* if it is a torsion coset of an abelian subvariety.
- $W \subset E^n$ is *weakly special* if it is a coset of an abelian subvariety.

Unlikely Intersections: Basic Setup

Specials in E^n

- $W \subset E^n$ is *special* if it is a torsion coset of an abelian subvariety.
- $W \subset E^n$ is *weakly special* if it is a coset of an abelian subvariety.

Specials in S^n

- $W \subset S^n$ is *special* if it is a connected Shimura subvariety.
- $W \subset S^n$ is *weakly special* if it is a connected Shimura Subvariety on some co-ordinates, constant on others.

Unlikely Intersections: Basic Setup

Specials in E^n

- $W \subset E^n$ is *special* if it is a torsion coset of an abelian subvariety.
- $W \subset E^n$ is *weakly special* if it is a coset of an abelian subvariety.

Specials in S^n

- $W \subset S^n$ is *special* if it is a connected Shimura subvariety.
- $W \subset S^n$ is *weakly special* if it is a connected Shimura Subvariety on some co-ordinates, constant on others.
- **Note: All weakly specials are defined by pairwise-relations.**

Unlikely Intersections: Basic Setup

Specials in E^n

- $W \subset E^n$ is *special* if it is a torsion coset of an abelian subvariety.
- $W \subset E^n$ is *weakly special* if it is a coset of an abelian subvariety.

Specials in S^n

- $W \subset S^n$ is *special* if it is a connected Shimura subvariety.
- $W \subset S^n$ is *weakly special* if it is a connected Shimura Subvariety on some co-ordinates, constant on others.
- **Note: All weakly specials are defined by pairwise-relations.**

A subvariety $W \subset S^n \times E^n$ is *(weakly) special* if it is a product $W = W_1 \times W_2$ of (weakly) specials.

Unlikely Intersections: Basic Setup

Specials in E^n

- $W \subset E^n$ is *special* if it is a torsion coset of an abelian subvariety.
- $W \subset E^n$ is *weakly special* if it is a coset of an abelian subvariety.

Specials in S^n

- $W \subset S^n$ is *special* if it is a connected Shimura subvariety.
- $W \subset S^n$ is *weakly special* if it is a connected Shimura Subvariety on some co-ordinates, constant on others.
- **Note: All weakly specials are defined by pairwise-relations.**

A subvariety $W \subset S^n \times E^n$ is *(weakly) special* if it is a product $W = W_1 \times W_2$ of (weakly) specials.

Unlikely Intersections: Zilber-Pink

- $M = S^n, E^n$ or $S^n \times E^n$. - *Mixed Shimura Variety*.

Unlikely Intersections: Zilber-Pink

- $M = S^n, E^n$ or $S^n \times E^n$. - *Mixed Shimura Variety*.
- $\langle Z \rangle$ - smallest special subvariety containing Z.
- $\delta(Z) := \dim\langle Z \rangle - \dim Z$. The *defect* of Z.

Unlikely Intersections: Zilber-Pink

- $M = S^n, E^n$ or $S^n \times E^n$. - *Mixed Shimura Variety*.
- $\langle Z \rangle$ - smallest special subvariety containing Z.
- $\delta(Z) := \dim\langle Z \rangle - \dim Z$. The *defect* of Z.
- If $A \subset Z$ has smaller defect, then $\dim(\langle A \rangle \cap Z) > \dim Z + \dim\langle A \rangle - \dim M$.

Unlikely Intersections: Zilber-Pink

- $M = S^n, E^n$ or $S^n \times E^n$. - *Mixed Shimura Variety*.
- $\langle Z \rangle$ - smallest special subvariety containing Z.
- $\delta(Z) := \dim\langle Z \rangle - \dim Z$. The *defect* of Z.
- If $A \subset Z$ has smaller defect, then $\dim(\langle A \rangle \cap Z) > \dim Z + \dim\langle A \rangle - \dim M$.
- $A \subset Z$ is *optimal* if $A \subset B \subset Z, \delta(B) \leq \delta(A) \Rightarrow A = B$.

Conjecture (Zilber-Pink)

$Z \subset M$ has finitely many optimal subvarieties.

Unlikely Intersections: Zilber-Pink

- $M = S^n, E^n$ or $S^n \times E^n$. - *Mixed Shimura Variety*.
- $\langle Z \rangle$ - smallest special subvariety containing Z.
- $\delta(Z) := \dim\langle Z \rangle - \dim Z$. The *defect* of Z.
- If $A \subset Z$ has smaller defect, then $\dim(\langle A \rangle \cap Z) > \dim Z + \dim\langle A \rangle - \dim M$.
- $A \subset Z$ is *optimal* if $A \subset B \subset Z, \delta(B) \leq \delta(A) \Rightarrow A = B$.

Conjecture (Zilber-Pink)

$Z \subset M$ has finitely many optimal subvarieties.

- ZP $\Rightarrow$ AO, special subvariety $\Leftrightarrow$ defect 0.
- We cannot prove full Zilber-Pink, even for $V^n \subset S^n \times E^n$.

Unlikely Intersections: Exemplary Subvarieties

- **Motivation:** Suppose $(\vec{s}, \vec{e}) \in V^n$ such that $\vec{s} \in \mathrm{CM}, \vec{e}$ are linearly dependent.

Unlikely Intersections: Exemplary Subvarieties

- **Motivation:** Suppose $(\vec{s}, \vec{e}) \in V^n$ such that $\vec{s} \in \mathrm{CM}, \vec{e}$ are linearly dependent. Then $\langle \vec{e} \rangle = B \subsetneq E^n$, $\langle (\vec{s}, \vec{e}) \rangle = \vec{s} \times B$ is atypical.

Unlikely Intersections: Exemplary Subvarieties

- **Motivation:** Suppose $(\vec{s}, \vec{e}) \in V^n$ such that $\vec{s} \in \mathrm{CM}, \vec{e}$ are linearly dependent. Then $\langle \vec{e} \rangle = B \subsetneq E^n$, $\langle (\vec{s}, \vec{e}) \rangle = \vec{s} \times B$ is atypical.
- $W \subset V^n$ is *exemplary* if
  - $\pi_{S^n}(W)$ is special
  - $\langle \pi_{E^n}(W) \rangle = B \subsetneq E^n$
  - Given the above, $\pi_{S^n}(W)$ is maximal.

Unlikely Intersections: Exemplary Subvarieties

- **Motivation:** Suppose $(\vec{s}, \vec{e}) \in V^n$ such that $\vec{s} \in \mathrm{CM}$, $\vec{e}$ are linearly dependent. Then $\langle \vec{e} \rangle = B \subsetneq E^n$, $\langle (\vec{s}, \vec{e}) \rangle = \vec{s} \times B$ is atypical.
- $W \subset V^n$ is *exemplary* if
  - $\pi_{S^n}(W)$ is special
  - $\langle \pi_{E^n}(W) \rangle = B \subsetneq E^n$
  - Given the above, $\pi_{S^n}(W)$ is maximal.

Theorem (Main II, Pila-T)

There are finitely many exemplary subvarieties of V^n.

Unlikely Intersections: Exemplary Subvarieties

- **Motivation:** Suppose $(\vec{s}, \vec{e}) \in V^n$ such that $\vec{s} \in \mathrm{CM}, \vec{e}$ are linearly dependent. Then $\langle \vec{e} \rangle = B \subsetneq E^n$, $\langle (\vec{s}, \vec{e}) \rangle = \vec{s} \times B$ is atypical.
- $W \subset V^n$ is *exemplary* if
  - $\pi_{S^n}(W)$ is special
  - $\langle \pi_{E^n}(W) \rangle = B \subsetneq E^n$
  - Given the above, $\pi_{S^n}(W)$ is maximal.

Theorem (Main II, Pila-T)

There are finitely many exemplary subvarieties of V^n.

Main II $\Rightarrow$ Main I. Indeed, for any special $Z \in S^n, \exists D > 0, i, j$
$\vec{s} \in Z \Rightarrow s_i, s_j$ are D-dependent.

Unlikely Intersections: Exemplary Subvarieties

- **Motivation:** Suppose $(\vec{s}, \vec{e}) \in V^n$ such that $\vec{s} \in \mathrm{CM}, \vec{e}$ are linearly dependent. Then $\langle \vec{e} \rangle = B \subsetneq E^n$, $\langle (\vec{s}, \vec{e}) \rangle = \vec{s} \times B$ is atypical.
- $W \subset V^n$ is *exemplary* if
    - $\pi_{S^n}(W)$ is special
    - $\langle \pi_{E^n}(W) \rangle = B \subsetneq E^n$
    - Given the above, $\pi_{S^n}(W)$ is maximal.

Theorem (Main II, Pila-T)

There are finitely many exemplary subvarieties of V^n.

Main II $\Rightarrow$ Main I. Indeed, for any special $Z \in S^n, \exists D > 0, i, j$
$\vec{s} \in Z \Rightarrow s_i, s_j$ are D-dependent.
Exemplary special subvarieties are optimal, so ZP $\Rightarrow$ Main II.

Ingredients: Functional Transcendence

Consider $\pi : \mathbb{H}^n \times \mathbb{C}^n \to S^n \times E^n$.

Theorem (Z.Gao, Weak Ax-Schanuel)

Let $A \subset \mathbb{H}^n \times \mathbb{C}^n, B \subset S^n \times E^n$ *be (semi)- algebraic.*
$\dim(A \cap B)^\circ > \dim A + \dim B - 2n \Rightarrow \pi((A \cap B)^\circ)$ *is contained in a proper weakly special.*

Ingredients: Functional Transcendence

Consider $\pi : \mathbb{H}^n \times \mathbb{C}^n \to S^n \times E^n$.

Theorem (Z.Gao, Weak Ax-Schanuel)

Let $A \subset \mathbb{H}^n \times \mathbb{C}^n, B \subset S^n \times E^n$ be (semi)- algebraic.
$\dim(A \cap B)^\circ > \dim A + \dim B - 2n \Rightarrow \pi((A \cap B)^\circ)$ *is contained in a proper weakly special.*

Informally: The only way to pass algebraic information between $\mathbb{H}^n \times \mathbb{C}^n$ and $S^n \times E^n$ is via weakly-specials.

Ingredients: o-minimality

We work with the o-minimal structure $\mathbb{R}_{an,exp}$:

Ingredients: o-minimality

We work with the o-minimal structure $\mathbb{R}_{an,exp}$:

- This is a theory of 'tame geometry'

Ingredients: o-minimality

We work with the o-minimal structure $\mathbb{R}_{an,exp}$:

- This is a theory of 'tame geometry'
- X is *definable* in $\mathbb{R}_{an,exp}$ if it can be built up of polynomials, $y = e^x$, overconvergent analytic functions, boolean operations, and projections.

Ingredients: o-minimality

We work with the o-minimal structure $\mathbb{R}_{an,exp}$:

- This is a theory of 'tame geometry'
- X is *definable* in $\mathbb{R}_{an,exp}$ if it can be built up of polynomials, $y = e^x$, overconvergent analytic functions, boolean operations, and projections.
- Definable sets have a nice dimension theory, cell decomposition, finiteness properties, etc...
- For $\mathcal{F} \subset \mathbb{H} \times \mathbb{C}$ the usual fundamental domain, $\pi : \mathcal{F} \to S \times E$ is definable.

Ingredients: o-minimality

We work with the o-minimal structure $\mathbb{R}_{an,exp}$:

- This is a theory of 'tame geometry'
- X is *definable* in $\mathbb{R}_{an,exp}$ if it can be built up of polynomials, $y = e^x$, overconvergent analytic functions, boolean operations, and projections.
- Definable sets have a nice dimension theory, cell decomposition, finiteness properties, etc...
- For $\mathcal{F} \subset \mathbb{H} \times \mathbb{C}$ the usual fundamental domain, $\pi : \mathcal{F} \to S \times E$ is definable.

Theorem (Pila-Wilkie,2004)

If $X \subset \mathbb{R}^n$ is definable, and does not contain any semi-algebraic curves, then

$$\#\{x \in X \cap \mathbb{Z}^n, H(x) < T\} = T^{o(1)}$$

.

Proof of Main II: Finitely many slopes

We first handle the case where S is *strongly special*, so that no co-ordinate is constant.

Proof of Main II: Finitely many slopes

We first handle the case where S is *strongly special*, so that no co-ordinate is constant.

- *Lin* - linear subvarieties in $\mathbb{C}^n$.
- *Mob* - strongly Mobius subvarieties in $\mathbb{H}^n$ (relations of the form $z_i = gz_j$, no constant co-ordinates).

Proof of Main II: Finitely many slopes

We first handle the case where S is *strongly special*, so that no co-ordinate is constant.

- *Lin* - linear subvarieties in $\mathbb{C}^n$.
- *Mob* - strongly Mobius subvarieties in $\mathbb{H}^n$ (relations of the form $z_i = gz_j$, no constant co-ordinates).
- For $(L, G) \in Lin \times Mob$ $K(L, G) := (L \times G) \cap \pi^{-1}(V^n) \cap \mathcal{F}^n$.

Proof of Main II: Finitely many slopes

We first handle the case where S is *strongly special*, so that no co-ordinate is constant.

- *Lin* - linear subvarieties in $\mathbb{C}^n$.
- *Mob* - strongly Mobius subvarieties in $\mathbb{H}^n$ (relations of the form $z_i = gz_j$, no constant co-ordinates).
- For $(L, G) \in Lin \times Mob$ $K(L, G) := (L \times G) \cap \pi^{-1}(V^n) \cap \mathcal{F}^n$.
- If $\dim K(L, G) = L$ then AO$\Rightarrow \exists L_0 \subset L, G_0 \subset G$ weakly special, and $\dim K(L_0, G_0) = L_0$.

Proof of Main II: Finitely many slopes

We first handle the case where S is *strongly special*, so that no co-ordinate is constant.

- *Lin* - linear subvarieties in $\mathbb{C}^n$.
- *Mob* - strongly Mobius subvarieties in $\mathbb{H}^n$ (relations of the form $z_i = gz_j$, no constant co-ordinates).
- For $(L, G) \in Lin \times Mob$ $K(L, G) := (L \times G) \cap \pi^{-1}(V^n) \cap \mathcal{F}^n$.
- If $\dim K(L, G) = L$ then AO$\Rightarrow \exists L_0 \subset L, G_0 \subset G$ weakly special, and $\dim K(L_0, G_0) = L_0$.
- In fact, $L \in Mob \Rightarrow L$ is special.
- $R :=$ $\{(L, G) \mid \ \dim K(L, G) = \ \dim L,\ L \text{ maximal },\ G \text{ minimal}\}$.

Proof of Main II: Finitely many slopes

We first handle the case where S is *strongly special*, so that no co-ordinate is constant.

- *Lin* - linear subvarieties in $\mathbb{C}^n$.
- *Mob* - strongly Mobius subvarieties in $\mathbb{H}^n$ (relations of the form $z_i = gz_j$, no constant co-ordinates).
- For $(L, G) \in Lin \times Mob$ $K(L, G) := (L \times G) \cap \pi^{-1}(V^n) \cap \mathcal{F}^n$.
- If $\dim K(L, G) = L$ then AO$\Rightarrow \exists L_0 \subset L, G_0 \subset G$ weakly special, and $\dim K(L_0, G_0) = L_0$.
- In fact, $L \in Mob \Rightarrow L$ is special.
- $R :=$ $\{(L, G) \mid \ \dim K(L, G) = \ \dim L, \ L \text{ maximal }, \ G \text{ minimal}\}$.
- R is discrete and definable $\Rightarrow R$ is finite.

Proof of Main II: Finitely many slopes

We first handle the case where S is *strongly special*, so that no co-ordinate is constant.

- Lin - linear subvarieties in $\mathbb{C}^n$.
- Mob - strongly Mobius subvarieties in $\mathbb{H}^n$ (relations of the form $z_i = gz_j$, no constant co-ordinates).
- For $(L, G) \in Lin \times Mob$ $K(L, G) := (L \times G) \cap \pi^{-1}(V^n) \cap \mathcal{F}^n$.
- If $\dim K(L, G) = L$ then AO$\Rightarrow \exists L_0 \subset L, G_0 \subset G$ weakly special, and $\dim K(L_0, G_0) = L_0$.
- In fact, $L \in Mob \Rightarrow L$ is special.
- $R :=$ $\{(L, G) \mid \ \dim K(L, G) = \ \dim L, \ L \text{ maximal }, \ G \text{ minimal}\}$.
- R is discrete and definable $\Rightarrow$ R is finite.

Proof of Main II: Isolated CM points

We now handle the case where $S = \vec{s}$ is a single CM points.

Proof of Main II: Isolated CM points

We now handle the case where $S = \vec{s}$ is a single CM points.

- Suppose $\vec{x}$ is a V-images of s, and $\sum_i n_i x_i = 0$.

Proof of Main II: Isolated CM points

We now handle the case where $S = \vec{s}$ is a single CM points.

- Suppose $\vec{x}$ is a V-images of s, and $\sum_i n_i x_i = 0$.
- $\Delta := \max_i(\Delta(s_i))$.
- $\max_i([\mathbb{Q}(x_i) : \mathbb{Q}], h(x_i)) \ll \Delta^{O(1)}$
- (Masser) We can choose $n_i \ll \Delta^{O(1)}$.

Proof of Main II: Isolated CM points

We now handle the case where $S = \vec{s}$ is a single CM points.

- Suppose $\vec{x}$ is a V-images of s, and $\sum_i n_i x_i = 0$.
- $\Delta := \max_i(\Delta(s_i))$.
- $\max_i([\mathbb{Q}(x_i) : \mathbb{Q}], h(x_i)) \ll \Delta^{O(1)}$
- (Masser) We can choose $n_i \ll \Delta^{O(1)}$.
- (Siegel, T.)$\exists\delta > 0, \max_i([\mathbb{Q}(s_i) : \mathbb{Q}]) \gg \Delta^{\delta}$.

Proof of Main II: Isolated CM points

We now handle the case where $S = \vec{s}$ is a single CM points.

- Suppose $\vec{x}$ is a V-images of s, and $\sum_i n_i x_i = 0$.
- $\Delta := \max_i(\Delta(s_i))$.
- $\max_i([\mathbb{Q}(x_i) : \mathbb{Q}], h(x_i)) \ll \Delta^{O(1)}$
- (Masser) We can choose $n_i \ll \Delta^{O(1)}$.
- (Siegel, T.)$\exists \delta > 0, \max_i([\mathbb{Q}(s_i) : \mathbb{Q}]) \gg \Delta^{\delta}$.
- $R := \{\vec{\tau} \in \mathcal{F}_{\mathbb{H}}^n \mid \exists \vec{r} \in \mathcal{F}_{\mathbb{C}}^n, \pi(\vec{\tau}, \vec{r}) \in V^n, \sum_i n_i \pi(r_i) = 0\}$.

Proof of Main II: Isolated CM points

We now handle the case where $S = \vec{s}$ is a single CM points.

- Suppose $\vec{x}$ is a V-images of s, and $\sum_i n_i x_i = 0$.
- $\Delta := \max_i(\Delta(s_i))$.
- $\max_i([\mathbb{Q}(x_i) : \mathbb{Q}], h(x_i)) \ll \Delta^{O(1)}$
- (Masser) We can choose $n_i \ll \Delta^{O(1)}$.
- (Siegel, T.)$\exists \delta > 0, \max_i([\mathbb{Q}(s_i) : \mathbb{Q}]) \gg \Delta^{\delta}$.
- $R := \{\vec{\tau} \in \mathcal{F}_{\mathbb{H}}^n \mid \exists \vec{r} \in \mathcal{F}_{\mathbb{C}}^n, \pi(\vec{\tau}, \vec{r}) \in V^n, \sum_i n_i \pi(r_i) = 0\}$.
- R definable, has many quadratic points of small height, PW$\Rightarrow$ R contains a semi-algebraic curve C.

Proof of Main II: Isolated CM points

We now handle the case where $S = \vec{s}$ is a single CM points.

- Suppose $\vec{x}$ is a V-images of s, and $\sum_i n_i x_i = 0$.
- $\Delta := \max_i(\Delta(s_i))$.
- $\max_i([\mathbb{Q}(x_i) : \mathbb{Q}], h(x_i)) \ll \Delta^{O(1)}$
- (Masser) We can choose $n_i \ll \Delta^{O(1)}$.
- (Siegel, T.)$\exists \delta > 0, \max_i([\mathbb{Q}(s_i) : \mathbb{Q}]) \gg \Delta^{\delta}$.
- $R := \{\vec{\tau} \in \mathcal{F}_{\mathbb{H}}^n \mid \exists \vec{r} \in \mathcal{F}_{\mathbb{C}}^n, \pi(\vec{\tau}, \vec{r}) \in V^n, \sum_i n_i \pi(r_i) = 0\}$.
- R definable, has many quadratic points of small height, PW$\Rightarrow$ R contains a semi-algebraic curve C.
- AO$\Rightarrow$ $\pi(C)$ is contained in a weakly special S'. But $S \in S' \Rightarrow S'$ is special.
- But then S is not maximal. Contradiction.

Proof of Main II: Isolated CM points

We now handle the case where $S = \vec{s}$ is a single CM points.

- Suppose $\vec{x}$ is a V-images of s, and $\sum_i n_i x_i = 0$.
- $\Delta := \max_i(\Delta(s_i))$.
- $\max_i([\mathbb{Q}(x_i):\mathbb{Q}], h(x_i)) \ll \Delta^{O(1)}$
- (Masser) We can choose $n_i \ll \Delta^{O(1)}$.
- (Siegel, T.)$\exists\delta > 0, \max_i([\mathbb{Q}(s_i):\mathbb{Q}]) \gg \Delta^{\delta}$.
- $R := \{\vec{\tau} \in \mathcal{F}_{\mathbb{H}}^n \mid \exists \vec{r} \in \mathcal{F}_{\mathbb{C}}^n, \pi(\vec{\tau}, \vec{r}) \in V^n, \sum_i n_i \pi(r_i) = 0\}$.
- R definable, has many quadratic points of small height, PW$\Rightarrow$ R contains a semi-algebraic curve C.
- AO$\Rightarrow$ $\pi(C)$ is contained in a weakly special S'. But $S \in S' \Rightarrow S'$ is special.
- But then S is not maximal. Contradiction.

Going from $E/\overline{\mathbb{Q}}$ to $E/\mathbb{C}$ is a specialization argument + Controlling complexity of special subvarieties in families using o-minimality.

Open Questions

- Zilber-Pink is still open, even for $X(1)^n$. *Partial Results by Habegger-Pila*

Open Questions

- Zilber-Pink is still open, even for $X(1)^n$. *Partial Results by Habegger-Pila*
- Allow questions of the type: Characterize triples $(x_1, x_2, x_3) \in S$ which are pairwise isogenous, whose image in E is linearly dependent.

Open Questions

- Zilber-Pink is still open, even for $X(1)^n$. *Partial Results by Habegger-Pila*
- Allow questions of the type: Characterize triples $(x_1, x_2, x_3) \in S$ which are pairwise isogenous, whose image in E is linearly dependent.
- Hard Part: obtaining lower bounds for Galois orbits of unlikely intersections. *Here we used Masser+Siegel.*

Thank you!