Effectivity in Faltings' Theorem.

Levent Alpoge

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- Obviously, we'd like to actually find $C(K)$ given C/K, i.e. produce a finite-time algorithm computing $(C, K) \mapsto C(K)$.

Theorem (A.-Lawrence).

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- (This is really a theorem about abelian varieties.)

Theorem (A.-Lawrence).
There is an algorithm that, on input (g, K, S) with $g \in \mathbb{Z}^+$, $K/\mathbb{Q}$ a number field, and S a finite set of places of K, outputs $\mathcal{A}_g(\mathfrak{o}_{K,S})$ ***if it terminates****.*

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- Our key argument closely follows (and indeed we depended quite a bit upon) the paper "Anabelian geometry and descent obstructions on moduli spaces" by Patrikis-Voloch-Zarhin.

Theorem (A.).

There is a finite-time algorithm that, on input $(\mathfrak{o}, K, S)$ *with* $\mathfrak{o}$ *an order in a totally real field,* $K/\mathbb{Q}$ *totally real with* $[K:\mathbb{Q}]$ *odd, and* S *a finite set of places of* K*, outputs* $\mathcal{H}_{\mathfrak{o}}(\mathfrak{o}_{K,S})$ *(i.e. the* A/K *good outside* S *and* $\mathrm{GL}_2(\mathfrak{o})$*-type over* K*).*

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- $K = \mathbb{Q}$ and $\mathfrak{o} = \mathbb{Z}$: Murty-Pasten. Independently, $K = \mathbb{Q}$: von Känel. Key point in his argument, which we build upon: Serre's conjecture is known over $\mathbb{Q}$, so the relevant abelian varieties are quotients of $J_0(N_S)$ for N_S explicit, now bound heights using Masser-Wüstholz/Bost.

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- Each $H_{\mathfrak{o}}$ contains lots of complete curves, thus giving unconditional algorithms for a class of curves over odd-degree totally real fields.
- We'll solve $x^6 + 4y^3 = 1$ (and some twists) over such K using a slightly modified argument.

Quick reminder: Faltings' first proof.

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- Thus by properness we reduce to dealing with $\mathcal{A}_g(\mathfrak{o}_{K,S})$, i.e. g-dimensional A/K with good reduction outside S.

$$\left\{ A/K : \begin{array}{c} \dim A = g, \\ N_A \ll_S 1 \end{array} \right\} \xrightarrow{/\text{isog.}} \left\{ \rho : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathrm{GSp}_{2g}(\mathbb{Q}_\ell) \right\} \hookrightarrow \mathbb{Z}^T$$

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(Or: "$L(s) \mapsto$ its first few Dirichlet coeff.s".)

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- The second map is easily injective ("Faltings' Lemma"). It is also *evident* that the image of the composition is finite! (Purity, aka Weil's proof of RH for curves: $a_\mathfrak{p} := \mathrm{tr}(\rho_{A,\ell}(\mathrm{Frob}_\mathfrak{p})) \in \mathbb{Z}$, and also $|a_\mathfrak{p}| \leq 2g \cdot \sqrt{\mathrm{Nm}\,\mathfrak{p}}$.)

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- So, just like in Vojta's proof, the ineffectivity arises because we do not know a priori whether a given $(a_\mathfrak{p})_{\mathfrak{p}\in T}$ has a corresponding A/K or not.

But it’s not hopeless…

- Well… for each n, $(a_{\mathfrak{p}})_{\mathfrak{p} \in T} \pmod{\ell^n}$ should at least be the traces of a rep. $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathrm{GSp}_{2g}(\mathbb{Z}/\ell^n)$ unram. outside S and the primes above ℓ.

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- That rep. should "look" pure of weight 1 (i.e. for $\mathrm{Nm}\,\mathfrak{p} \ll \ell^n$, $\mathrm{tr}(\rho(\mathrm{Frob}_{\mathfrak{p}})) \in \mathbb{Z}/\ell^n$ should be the reduction of an $a_{\mathfrak{p}} \in \mathbb{Z}$ with $|a_{\mathfrak{p}}| \leq 2g \cdot \sqrt{\mathrm{Nm}\,\mathfrak{p}}$) and have cyclotomic similitude character.

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- Each of these can at least be checked in finite time…

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- Is this enough?
- Not quite, but almost…

- If an $(a_\mathfrak{p})_{\mathfrak{p}\in T}$ "survives forever", then (Kőnig’s Lemma) it at least arises from a $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$ which:

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- ($\rho^{\text{s.s.}}$ satisfies the same properties (exactness of $D_{\text{crys.}}$) so wlog ρ is semisimple.)

- So, being finitely ramified and de Rham at primes above ℓ, by Fontaine-Mazur + Grothendieck-Serre, it is a summand of $H^i(X, \mathbb{Q}_\ell)(j)$ with X/K smooth projective.

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- Restricting scalars, we end up with a motive over K with $\mathbb{Q}$-coefficients whose ℓ-adic realization admits ρ as a summand.

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- Restricting scalars, we end up with a motive over K with $\mathbb{Q}$-coefficients whose ℓ-adic realization admits ρ as a summand.
- Our knowledge of Hodge-Tate weights implies the corresponding Betti realization is a $\mathbb{Q}$-Hodge structure of weight $\{\underbrace{(1,0),\ldots,(1,0)}_{g\cdot[E:\mathbb{Q}]},\underbrace{(0,1),\ldots,(0,1)}_{g\cdot[E:\mathbb{Q}]}\}$.

- So, being finitely ramified and de Rham at primes above ℓ, by Fontaine-Mazur + Grothendieck-Serre, it is a summand of $H^i(X, \mathbb{Q}_\ell)(j)$ with X/K smooth projective.
- By Tate the corresponding projector is a $\mathbb{Q}_\ell$-linear combination of correspondences.
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- We've found an abelian variety! First over $\mathbb{C}$ (Riemann), then over $\overline{\mathbb{Q}}$ (Hodge, really absolute Hodge), then over K (restricting scalars).

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- Eventually (per the conjectures) we find such a B/K, and then if $(a_{\mathfrak{p}})_{\mathfrak{p}\in T}$ arises from an A/K, by Poincaré complete reducibility we have $B \sim_K A \times C$, and so $h(A) \overset{\text{Bost}}{\leq} h(A \times C) + O_{\dim B}(1) \overset{\text{Masser-Wüstholz}}{\ll}_{h(B),[K:\mathbb{Q}],\dim B} 1.$

And unconditionally…?

- Let now $K/\mathbb{Q}$ be totally real with $[K : \mathbb{Q}]$ odd. Let also $F/\mathbb{Q}$ be totally real, and S a finite set of places of K.

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- We'll get away with using potential modularity.

- Recall that being $\mathrm{GL}_2(F)$-type buys us λ-adic Tate modules, i.e. 2-dimensional rep.s $\rho_{A,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathrm{GL}_2(\mathfrak{o}_{F,\lambda})$.

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- So find a large λ, and then determine all possible $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathrm{GL}_2(\mathfrak{o}_F/\lambda)$ (unram. outside $\dots$, correct det., $\dots$) explicitly (Hermite-Minkowski).

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- Only not-completely-explicit step: invocation of Moret-Bailly's theorem, which guarantees a totally real point on a twist of a Hilbert modular variety. Well, you can just brute force once you know the point exists...

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  — Solvable descent applied to $L/L^{\text{2-Sylow}}$! :)

- So we get an explicit finite set of totally real L/K with $[L : \mathbb{Q}]$ odd such that any relevant A/K is modular over L, and then Masser-Wüstholz/Bost again.

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- Easy to compute an explicit $\mathcal{F}$ for which each A_P/K is $\mathrm{GL}_2(F)$-type for some $F \in \mathcal{F}$, so we're done.

Thanks!